QUATERNION ALGEBRA

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1 Introduction

The report have two parts. In the fist part, some concepts of quaternion is presented. In the second part, the operator $\varphi_a = aqa^*, a \in \mathbb{H}^1, q \in \mathbb{T}$ will be tried to interpret by geometrically approach but it is only my view on quaternion.

1.1 Addition and multiplication

The set of quaternions $\mathbb{H}$ equal to $\mathbb{R}^4$, a four-dimensional vector space over the real numbers:

$$\mathbb{H} = (a, b, c, d)|a, b, c, d \in \mathbb{R}.$$

A quaternion $q \in \mathbb{H}$ is defined in the form:

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}. \tag{1.1}$$

The set $\mathbb{H}$ has three operations: addition, scalar multiplication, and quaternion
multiplication.

For $q, q_1 \in \mathbb{H}$ and $r \in \mathbb{R}$,

$$q + q_1 = (a + a_1) + (b + b_1)\mathbf{i} + (c + c_1)\mathbf{j} + (d + d_1)\mathbf{k}. \tag{1.2}$$

and

$$rq = ra + rb\mathbf{i} + rc\mathbf{j} + rd\mathbf{k}. \tag{1.3}$$

Multiplication for the basis elements of $\mathbb{H}$ is defined by

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = -\mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{1.4}$$

Now we can give the product of two quaternions q and q_1:

$$\begin{aligned} qq_1 = (a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})&(a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) = \\ &= (aa_1 - bb_1 - cc_1 - dd_1) + \\ &+ (ab_1 + a_1b + cd_1 - dc_1)\mathbf{i} + \\ &+ (ac_1 + a_1c - bd_1 + db_1)\mathbf{j} + \\ &+ (ad_1 + a_1d + bc_1 - cb_1)\mathbf{k}. \end{aligned} \tag{1.5}$$

1.2 Conjugate, Norm, and Inverse

Let $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ be a quaternion.

The *conjugate* of q is defined by

$$q^* = (a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^* = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}. \tag{1.6}$$

and it has the following properties:

1. $(q^*)^* = q$,
2. $q + q^* = 2a$,
3. $qq^* = q^*q = a2 + b2 + c2 + d2$,
4. $(qq_1)^* = q_1^* q^*$.

The *norm* of q is defined by $|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$ and the norm is equal $\sqrt{qq^*}$. The norm of a product of quaternions satisfies the properties $|q| = |q^*|$ and $|qq_1| = |q||q_1|$.

The *multiplication inverse* of a quaternion q is denoted by q^{-1} and has the property $qq^{-1} = q^{-1}q = 1$. It is constructed as

$$q^{-1} = q^*/|q|^2. \tag{1.7}$$

where the division of a quaternion by a real-valued scalar is just componentwise division. The inverse operation satisfies the properties $(q^{-1})^{-1} = 1$ and $(qq_1)^{-1} = q_1^{-1}q^{-1}$.

1.3 Matrix form

Quaternion $q \in \mathbb{H}$ can also be represented by a element of $\mathbb{M}_2(\mathbb{C})$ that have the following form:

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \tag{1.8}$$

here $\alpha = a + ib, \beta = c + id$. The sum and product of two such matrices is again of this form, and the sum and product of the quaternions corresponds to the sum and product of such matrices. Namely,

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ -\bar{\beta_1} & \bar{\alpha_1} \end{pmatrix} = \begin{pmatrix} \alpha\alpha_1 - \beta\bar{\beta_1} & \alpha\beta_1 + \bar{\alpha_1}\beta \\ -(\bar{\alpha}\bar{\beta_1} + \bar{\beta}\alpha_1) & \bar{\alpha}\bar{\alpha_1} - \bar{\beta}\beta_1 \end{pmatrix}$$

here

$$\alpha\alpha_1 - \beta\bar{\beta}_1 = (aa_1 - bb_1 - cc_1 - dd_1) + \mathbf{i}(ab_1 + a_1b + cd_1 - dc_1)$$
$$\alpha\beta_1 + \bar{\alpha}_1\beta = (ac_1 + a_1c - bd_1 + db_1) + \mathbf{i}(ad_1 + a_1d + bc_1 - cb_1).$$

The basis elements
$1 = (1, 0, 0, 0), \mathbf{i} = (0, 1, 0, 0), \mathbf{j} = (0, 0, 1, 0), \mathbf{k} = (0, 0, 0, 1)$ of $\mathbb{H}$ is written in the form:

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \left(\begin{matrix} 0 \\ i \end{matrix} \right.$$

and these matrices satisfy the relation (1.4). Moreover

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \Rightarrow q^* = \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix} = \bar{q}^t$$

Now, we express the norm of quaternion by matrix

$$|q|^2 = qq^* = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix} = \begin{pmatrix} \alpha\bar{\alpha} + \beta\bar{\beta} & 0 \\ 0 & \alpha\bar{\alpha} + \beta\bar{\beta} \end{pmatrix}$$
$$= (|\alpha|^2 + |\beta|^2)E = (|\alpha|^2 + |\beta|^2)(1 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) = \det q \tag{1.9}$$

1.4 Quaternion Operator

In the section, we present both subsets of $\mathbb{H}$.

$$\mathbb{H}^1 = \{a \in H| \ |a| = 1\}.$$

Every elements of $\mathbb{H}$ is unitary matrix. Indeed
$|a|^2 = aa^* = 1 \Rightarrow a^* = a^{-1}$.

$\mathbb{T}$ is a set of quaternions whose real part is zero and can be defined by

$$\mathbb{T} = \{x \in H| x^* = -x\}.$$

For a fixed $a \in \mathbb{H}^1$, we define an operator on vectors $x \in \mathbb{T}$:

$$\varphi_a(x) = axa^*. \tag{1.10}$$

We shall give proofs of the following states relating to the operator φ_a:

1. $\varphi_a(x) \in \mathbb{T}$;

2. $\|\varphi_a(x)\| = |x|$;

3. $x_1, x_2 \in \mathbb{T}, r_1, r_2 \in \mathbb{R}, \varphi_a(r_1x_1 + r_2x_2) = r_1\varphi_a(x_1) + r_2\varphi_a(x_2)$.

Proof:

1. $\varphi_a(x) = axa^{-1} \in \mathbb{H}$ and
   $\varphi_a(x) = (axa^{-1})^* = ax^*a^{-1} = -axa^{-1} = \varphi_a(x) \Rightarrow \varphi_a(x) \in \mathbb{T}$.

2. $|\varphi_a(x)|^2 = axa^{-1}ax^*a^{-1} = axx^*a^{-1} = a|x|^2a^{^{-1}} = |x|^2$.

3. $\varphi_a(r_1x_1 + r_2x_2) = a(r_1x_1 + r_2x_2)a^* = a(r_1x_1)a^* + a(r_2x_2)a^* =$
   $r_1\varphi_a(x_1) + r_2\varphi_a(x_2)$.$\square$

2 Rotation Operator

In this section, we formulate the value of φ_a at vector $q \in \mathbb{T}$ and demonstrate an interpretation geometrically of φ_a for a given $a = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{i}$. We can write a in the form:

$$a = \cos\frac{\theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin\frac{\theta}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} + \sin\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} - \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \text{ and } a^* = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}.$$

Let be $q \in \mathbb{T}$ and $q = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ $(x, y, z \in \mathbb{R})$. Then we have

$$\varphi_a(q) = aqa^* =$$
$$\begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}\begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix}\begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{i\frac{\theta}{2}} ix & e^{i\frac{\theta}{2}}(y+iz) \\ e^{-i\frac{\theta}{2}}(-y+iz) & -e^{-i\frac{\theta}{2}} ix \end{pmatrix} \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix} =$$

$$= \begin{pmatrix} ix & e^{i\theta}(y+iz) \\ e^{-i\theta}(-y+iz) & -ix \end{pmatrix}.$$

$\varphi_a(q) =$

$$\begin{pmatrix} ix & y\cos\theta - z\sin\theta + i(y\sin\theta + z\cos\theta) \\ -y\cos\theta + z\sin\theta + i(y\sin\theta + z\cos\theta) & -ix \end{pmatrix}$$

So the operator $\varphi_a : \mathbb{T} \to \mathbb{T}$ can be determined by the following representation:

$$\varphi_a : q = (x, y, z) \to p = (x, y\cos\theta - z\sin\theta, y\sin\theta + z\cos\theta) \quad (2.1)$$

Now we consider a rotation by an angle θ clockwise about x as the axis of rotation, denoted R^x_θ. Suppose that $p = R^x_\theta(q)$ and $p, q \in \mathbb{R}^3$. Using Figure 1, we can easily drive the following formula:

$$x' = x$$
$$y' = |p|\cos(\theta + \alpha) = |q|\cos\alpha\cos\theta - |q|\sin\alpha\sin\theta = y\cos\theta - z\sin\theta$$
$$z' = |p|\sin(\theta + \alpha) = |q|\cos\alpha\sin\theta + |q|\sin\alpha\cos\theta = y\sin\theta + z\cos\theta.$$

Here we assume that **i**, **j**, **k** is correspond with unit vector of axis X, Y, Z respectively. The relation between the operator φ_a and the rotation transformation R_θ^x is formulated by the following theorem.

Theorem 1 (3)**.** *For any unit quaternion*

$$a = \cos\frac{\theta}{2}E + \sin\frac{\theta}{2}\boldsymbol{u}$$

and for vector $q \in \mathbb{R}^3$ *the action of the operator*

$$L_a(q) = aqa^*$$

on q *is equivalent to a rotation of the vector* q *through an angle* θ *about* $\boldsymbol{u}$ *as the axis of rotation. where* $\boldsymbol{u} \in \mathbb{R}^3$ *is an unit vector .*

Let us consider the operator $\varphi_{\alpha\beta}$ for $\forall\alpha,\beta \in \mathbb{H}^1$. Indeed,

$$\varphi_{\alpha\beta}(q) = \alpha\beta q(\alpha\beta)^* = \alpha\beta q\beta^*\alpha^* = \alpha\varphi_\beta(q)\alpha^* = \varphi_\alpha(\varphi_\beta(q)).$$

So $\varphi_{\alpha\beta}$ is equal to the composition $\varphi_\alpha \circ \varphi_\beta$ of the two operators. Now we intend to explain the following example by geometrically approach.

Example: Let $\alpha, \beta \in \mathbb{H}^1$, $\alpha = \cos\frac{\theta}{2}E + \sin\frac{\theta}{2}\mathbf{i}$ and $\beta = \cos\frac{\theta}{2}E + \sin\frac{\theta}{2}\mathbf{j}$. Let us determine the angle and the axis of rotation equivalent $\varphi_{\alpha\beta}$.

$$\alpha\beta = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} =$$

$$\begin{pmatrix} e^{i\frac{\theta}{2}}\cos\frac{\theta}{2} & e^{i\frac{\theta}{2}}\sin\frac{\theta}{2} \\ -e^{-i\frac{\theta}{2}}\sin\frac{\theta}{2} & e^{-i\frac{\theta}{2}}\cos\frac{\theta}{2} \end{pmatrix}$$

and $\alpha\beta \leftrightarrow (\cos^2\frac{\theta}{2}, \sin\frac{\theta}{2}\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\sin\frac{\theta}{2})$. We can represent $\alpha\beta$ in the form:

$\alpha\beta = cos\frac{\theta_1}{2}E + sin\frac{\theta_1}{2}\mathbf{u}$

where $\cos\frac{\theta_1}{2} = \cos^2\frac{\theta}{2}$ and $\mathbf{u} = \frac{1}{\sqrt{1+\cos^2\frac{\theta}{2}}}(\cos\frac{\theta}{2}, \cos\frac{\theta}{2}, \sin\frac{\theta}{2})$.

The axis of the rotation equivalent to $\varphi_{\alpha\beta}$ is defined by the vector $\mathbf{u}$ and the angle of rotation is θ_1. When $\theta = 90°$, $\varphi_{\alpha\beta}$ is equivalent to a rotation about an axis defined by $(1, 1, 1)$ through an angle $120°$.

Ашигласан ном

[1] Jack B. Kuipers, Quaternions and Rotation squances, Coral Press, Sofia 2000, pp 127-143.

[2] Yan-Bin Jia, Quaternions and Rotations,(Com S 477/577 Notes) 2010.