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Development of Arithmetical Thinking: Evaluation of Subject Matter Knowledge of Pre-Service Teachers in Order to Design the Appropriate Course

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Abstract One of the key courses in the mathematics teacher education program in Israel is arithmetic, which engages in contents which these pre-service mathematics teachers (PMTs) will later teach at school. Teaching arithmetic involves knowledge about the essence of the concept of "number" and the development thereof, calculation methods and strategies. properties of operations on different sets of numbers, as well as the properties of the numbers themselves. Hence, the question arises: how to educate PMTs in order to supplement their mathematical knowledge with the required components? The present study explored the development of arithmetic thinking among pre-service teachers intending to teach mathematics at elementary school. This was done by matching the van Hiele theory of the development of geometric thinking to arithmetic. Analysis of findings obtained both in the present study and in many studies of geometry teaching indicates that this approach to considering the learners' level of thinking development might lead to meaningful learning in arithmetic course for PMTs.

Keywords Prospective mathematics teachers (PMTs) · Mathematical knowledge for teaching · Development of mathematical thinking · Arithmetic

Introduction

Teacher education for elementary school in Israel is mostly disciplinarian, i.e., pre-service teachers who are awarded a teaching certificate will teach a specific discipline studied at elementary school. The same also applies to mathematics teachers for this age group. In addition to general pedagogical courses, the education of these pre-service teachers comprises the study of mathematics and the didactics which correspond to the mathematical content offered both in the elementary school curriculum

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and at a more advanced level, in order to create the wide infrastructure of mathematical knowledge that teachers require. Subject matter studies constitute about one third of all the 4-year pre-service education courses. This includes courses on arithmetic, geometry, solid geometry, elementary number theory, and elementary set theory to name a few. The above courses should provide pre-service teachers with an adequate and solid mathematical infrastructure and they do not deal with pedagogical content knowledge (PCK) and curricular content knowledge (CCK), at least not formally. On the whole, pre-service teachers study at least 52 credit points in their field of specialization, in this case—mathematics. One of the key courses in the mathematics teacher education program in Israel is arithmetic, which engages in contents which these pre-service mathematics teachers (PMTs) will later teach at school.

This is a mathematics course which deals with sets of numbers and operations defined on these sets. The PMTs have studied many of these topics previously at school, one way or another. The main objective of the course is for them to comprehend the relevant contents as well as revise and re-arrange the contents they already know.

In Israeli teacher education colleges, this course is studied in the first year of the program. This is usually a year-long course comprising 28–30 sessions of two academic hours each. Due to the short time allocated to this course, its lecturers face quite a complex dilemma. On the one hand, the course should be organized so that attending students acquire as much content as possible. On the other hand, the course should be taught so that students assimilate the contents and are able to implement them when teaching their pupils. Coping with this dilemma is not simple. One of the reasons for this is the small number of studies which have investigated changes in the students' knowledge as a result of attending the arithmetic course (Thanheiser, Browning, Edson, Kastberg & Lo, 2013). Another reason is the effect of the various ways of organizing and teaching the course. However, most studies which do investigate PMTs' knowledge of arithmetic indicate that this knowledge is far from sufficient (e.g. Blömeke, Suhl & Döhrmann, 2013; Guberman & Leikin, 2012; Isksal & Cakiroglu, 2011). The question thus arises: How should arithmetic courses be organized so that they respond to PMTs' competencies, allowing them to reach the desired level by the end of the course? The answer resides in the compliance between their current level and the level at which the teachers teach (Ausubel, 1968; Van Hiele, 1986). This evokes the need to provide teacher trainers with tools which enable identification of their students' level of thinking, in order to adapt the teaching accordingly.

Theoretical Background

The study of teachers' knowledge, especially that of mathematics teachers, is an internationally popular topic: "Teachers have been tested, studied, analyzed, lauded, and criticized" (Hill, Ball, Sleep & Lewis, 2007, p. 111). Mathematics teachers' knowledge has been described as complex and multifaceted (Livy & Vale, 2011). The researchers proposed frameworks of mathematical knowledge needed for teaching, such as domains of mathematical knowledge for teaching (Ball, Thames & Phelps, 2008), the knowledge quartet (Rowland, Huckstep & Thwaites, 2005), 3D model of mathematics teachers' knowledge (Leikin, 2006), the mediation model between pedagogical content knowledge and content knowledge (Baumert et al., 2010), and so on.

The aim of these frameworks is to answer the question: What mathematical knowledge do teachers need to be well prepared for their job?

One of the components found in all the suggested frameworks is, of course, the mathematics content knowledge necessary for teachers. Based on numerous studies and reviews (e.g. Ma, 1999; Ball, Hill & Bass, 2005; Sowder, 2007), knowledge of the mathematics content area can be described as knowledge of mathematics which every adult and educated person has. In addition to this kind of knowledge, Ball, Thames, and Phelps described the specialized mathematics content knowledge that is unique to teaching (Ball, Thames & Phelps, 2008). Ma defined this knowledge as deep knowledge of mathematics at the level of profound understanding in fundamental mathematics (PUFM)—"understanding the terrain of fundamental mathematics that is deep, broad, and thorough" (Ma, 1999, p. 120). Understanding mathematics in depth means: knowing "what" and knowing "why." In the case of arithmetic, it means knowledge about the essence of the concept of "number" and the development thereof, calculation methods and strategies, properties of operations on different sets of numbers, as well as the properties of the numbers themselves. Hence, the question arises: how to educate PMTs in order to supplement their mathematical knowledge with the required components?

Development of Mathematical Knowledge for Teaching

Even though the process of developing mathematics teachers' knowledge is very important to the education of pre-service teachers, very little research has examined this issue (Lannin, Webb, Chval, Arbaugh, Hicks, Taylor & Bruton, 2013; Guberman, 2008).

Studies which do relate to the development of PMTs' mathematical knowledge and which are relevant to this study can be classified into two main groups. One group relates to the quality of the knowledge acquired by pre-service teachers and the second group is associated with the nature of the organization of mathematics courses designed for this population (Verschaffel, Greer & De Corte, 2007).

The first aspect to which researchers relate to is the knowledge acquired by the PMTs during their training and its manifestation in their teaching at school. These studies indicate that PMTs' mathematics content knowledge is insufficient (e.g. Ball, 1990; Blömeke et al., 2013; Livy & Vale, 2011; Saxe, Gearhart & Nasir, 2001). Moreover, they illustrate that although pre-service teachers acquire a certain mathematics content knowledge, they do not know how to use their knowledge to help learners understand the topics studied (Borko et al., 1992). For example, one of the possible reasons for pupils' difficulties in understanding fractions could be the difficulty their teachers have with this topic (Saxe, Gearhart & Nasir, 2001). Another study (Tirosh, 2000) found that PMTs join courses dealing with mathematics content knowledge equipped with prior knowledge that is mainly procedural. The procedural knowledge results from the kind of mathematical experience which the PMTs underwent during their studies at school and in most academic mathematics courses. Typically, the lesson starts with checking the homework, continues with presentation of new material, and the solution of tasks associated with this new material. It then ends with an exercise built upon problems similar to those presented immediately after the presentation (Sowder, 2007).

These and other studies lead to the conclusion that many teachers lack a conceptual understanding of elementary mathematics (Mewborn, 2001). At the same time, the PMTs don't have the opportunity to learn mathematics (or in this case—arithmetic) conceptually anywhere else in their teacher education programs (Thanheiser et al., 2013).

The second type of studies investigates how to plan a course aimed at developing PMTs' mathematics knowledge. One opinion is that such a course should be built like any other mathematics course in an academic discipline. Planning a course in this way facilitates organization of mathematics knowledge prevalent in the area of knowledge: all the concepts and principles associated with the course topic are well defined, the proofs are well structured, and all the items of knowledge are logically interconnected (Tatto et al., 2012). Conversely, Moreira & David (2008) argue that mathematics knowledge inculcated "in an academic manner" does not necessarily enhance the knowledge required by mathematics teachers. Academic knowledge might undermine the quality of mathematical-pedagogical knowledge, which teachers need. Nathan & Petrosino (2003) support this opinion and even indicate in their studies that PMTs with more advanced and formal knowledge tend to use mathematical principles and their accepted development method as a basis for guiding their teaching rather than learners' needs and mathematical development. This debate leads to the question whether mathematics courses for PMTs should be taught by lecturers from the departments of mathematics or by lecturers from schools or faculties of education (e.g. Baumert et al., 2010).

Furthermore, Artzt, Sultan, Curcio & Gurl (2012) suggest that pedagogical aspects and materials related to mathematics teaching at school, such as mathematical diaries of learners which include their perceptions of topics relevant to the course should be integrated into disciplinary courses. These researchers argue that a course engaging in mathematics contents required by teachers should include a mathematical topic given by the lecturer. The course should also deal with the solution of problems in multiple ways as well as with reference to different representations, including representations by means of technological tools. This suggestion is reinforced in view of assertions by Rowland et al., (2005), who maintain that similarly to mathematics lessons in which mathematics knowledge and pedagogical knowledge are manifested together, the development of these components among PMTs should be closely connected.

Hiebert & Lefevre (1986) maintain that mathematics courses designed for pre-service teachers should deal with conceptual understanding of mathematics, i.e., include rich knowledge of the connectivity between the facts, definitions, and properties of mathematical objects. The indication that something has been understood is the ability to think with what you know. Understanding is acquired through activities which necessitate such ways of thinking (Salomon & Perkins, 1996). In other words, the objective is to nurture the mathematical thinking of pre-service teachers in order to attain the necessary comprehension. Hence, a method should be sought to understand the development of mathematical thinking or, in the specific case of this study, a theory which explains the development of arithmetic thinking.

Battista claims that the van Hiele theory describes *a progression of mathematical thinking* and has as sufficient "ring of validity" also for researchers, curriculum developers, and teachers (Battista, 2007, p. 856). In the present study, I examined the impact of the arithmetic course on PMTs' knowledge, according to the van Hiele theory of development of mathematical thinking.

Development of Mathematical Thinking

In 1957, Dina and Pierre van Hiele presented their Ph.D. thesis in which they discussed the theory of the development of geometric thinking. The theory attempted to explain the fact that many pupils encounter difficulties in higher-order cognitive processes, particularly when they have to provide proofs. The goal of the van Hieles was to classify learners' attainments by levels of geometric thinking development. According to this theory, geometric thinking development can be viewed as a 5-level hierarchy. Some years later, Pierre Van Hiele reduced the number of levels of geometric thinking development to three (Van Hiele, 1999), believing that only few pupils progress from the third to the fourth level. In recent years, though, it has become common to work with four levels. The level descriptions presented here are based on Battista (2007).

Level 1: *Visualization*—the pupils can identify and work with geometric shapes according to their look: the shape is perceived as a whole (no attention is paid to its components) as it is seen.
Level 2: *Analysis*—the pupils identify and can characterize geometric shapes according to their attributes.
Level 3: *Abstraction*—the pupils understand the logical order of the shapes according to the attributes, the relations between the shapes and their attributes, as well as the importance of accurate definitions.
Level 4: *Formal deduction*—pupils are able to prove theorems within the same axiomatic system.

Van Hiele (1986) maintains that the progress from one level to another is parallel to the development of language. That is, one can identify the learners' level of thinking development through the arguments or explanations they provide.

The van Hiele theory was presented by Wirszup (1976) to the mathematics educators' community in the USA. Since then, numerous researchers have investigated and validated this theory. Some researchers (e.g. Usiskin, 1982) developed tools for identifying and assessing the levels among various learner populations. Others (e.g. Crowley, 1990) wanted to determine and analyze characteristics of geometric thinking levels. Yet others (e.g. Clements, Battista & Sarama, 2001) developed teaching units grounded in this theory. These joint efforts led to a valid tool for teachers and curriculum developers, but which dealt only in plane geometry.

In his book *Structure and Insight*: *A Theory of Mathematics Education*, Van Hiele (1986) argued that the levels of thinking are not limited solely to geometry. He maintained that they can be applied to any mathematical area studied at school and gave general descriptions typical of mathematical thinking. From this derived the idea to apply this approach to arithmetic, using a theory conceived for a more effective structure of a course dealing with the teaching of arithmetic contents to students (Guberman, 2008). The present study aims, therefore, to determine the levels of arithmetic thinking of elementary school PMTs at the beginning and end of the arithmetic course. The objective of this is to suggest a way to design an appropriate arithmetic course for pre-service teachers using an adaptation of van Hiele's model in order to match the level of teaching with the level of the PMTs' thinking.

Methodology

Research Population

The subjects of this study were students in four Israeli academic teacher education colleges in the programs for elementary school mathematics teachers. The students were in their first year of the education program. At the beginning of the year, 96 students responded to a pre-course questionnaire. At end of the course, 94 of these responded to the same questionnaire. It is worth noting that students specializing in elementary school mathematics teaching do not constitute a large population in Israel. The research sample includes a considerable segment of this population.

Research Tools

The main research tool was a 20-item questionnaire designed to determine pre-service teachers' level of arithmetic thinking. In order to construct this questionnaire, I performed an a priori match between the levels of geometric thinking development according to Van Hiele, and arithmetic thinking levels. Then I looked for assignments which facilitate testing of the various levels of thinking. Some of the assignments were taken from various sources and others were designed by me. A total of 20 mathematical assignments were formulated, inducing answers which match the four levels of thinking (see Appendix 1).

For the purpose of the present study, the mathematical assignments were chosen in accordance with several principles:

1. Questions with answers illustrating basic competencies.
2. Questions of moderate technical difficulty.
3. Questions relevant to the elementary school curriculum. It is important to point out that the elementary school curriculum does not attest that pupils are at a specific level. It is rather an observation whereby the questionnaire can include items relating to subjects familiar to the students. Otherwise, the wrong answer to one of the items might have resulted from lack of knowledge of the subject and would not have illustrated the level of thinking. For example, the notion of ratio in the elementary school curriculum is not sufficiently studied (approximately 8 h in Israeli syllabus) and the arithmetic aspects of this subject are not addressed in higher grades. Consequently, the questionnaire does not include items relating to ratio.
4. Questions describing non-standard situations as a component of a familiar process of solution.

In order to determine the level of arithmetic thinking development, I decided to use multi-choice items based on the model built by Usiskin (1982). Accordingly, all the questions were presented as multiple-choice tasks. After construction of the first version of the questionnaire, a pilot study was conducted among six students specializing in elementary school mathematics teaching. This pilot study consisted of a questionnaire filled in by students who had already completed the arithmetic course. Each student demonstrated a different level of attainment. This fact made it possible to assume that they represented different levels of arithmetic thinking development.

Following the pilot study, verbal and logical-semantic changes were made in those questions which were not clear to the students. Furthermore, it was decided at that stage to ask the students to explain the solution of the entire question, due to the known shortcoming of multiple-choice items (the examiner's inability to know why a specific option was chosen). Moreover, analysis of these reasons facilitated characterization of the levels of arithmetic thinking development and enhanced the types of answer suitable to each level.

The final questionnaire comprised 20 items. Each level of thinking comprised five items. Students who responded correctly to at least four out of the five items representing a certain level and were well-versed in all the previous levels were defined as "having mastered" this level.

This questionnaire was validated in three ways: the pilot study, expert judgment and the Guttman Scale Analysis. Experts were asked to review test items to determine if each is representative and relevant to the domain being measured. The Rep coefficient (Guttman Scale Analysis) for a pre-test is 0.94; the Rep coefficient for a post-test is 0.956. One can therefore argue with a high degree of probability that the questionnaire constructed defines a hierarchical structure.

Research Procedure

The data collection for the present study was conducted in two stages. At the beginning of the academic year, the preliminary questionnaire was administered to all the students who participated in the study. The questionnaire was administered in each of the colleges during the first 2 weeks of the academic year. Questionnaire completion lasted 1 h. Toward the end of the academic year, some 3 weeks before the end of the second semester, the same questionnaire was administered to the students.

Data Analysis

As mentioned earlier, each questionnaire item was a multiple-choice question and its solution included choice of the appropriate answer and justification of that choice. Thus, the data obtained were analyzed both quantitatively and qualitatively.

The quantitative analysis of the questionnaire results included:

- Calculation of the frequency of arithmetic thinking levels of the PMTs
- A χ^2 test checking the significance of changes in the levels of thinking of students in all the colleges, after attending the arithmetic course

The Qualitative Analysis of Explanations from the Questionnaire

The explanations provided for each of the answers were collected from all the questionnaires. Out of all the explanations given to a specific question, the explanations given by the students who are at the level of the question were extracted. Assignation to a level was based on results of the prior quantitative analysis. Thus, for example, among all the respondents to item 1, only the explanations of respondents designated as being at the first level were analyzed. The third stage of analysis involved characterization of four categories of explanations based on the types of answers given by learners who were at the various levels of arithmetic thinking. This involved checking and revising types determined a priori by analyzing the explanations given to the items at that level. The main role of these answer types was to constitute a tool to identify the thinking level of the learners whose answers belong to a certain category. The following is an example of item 5 that was presented to the students in this study. The students' explanations can be classified into three main groups:

- Students who multiplied 5.5 by 3.2 (some students used an algorithm for multiplying decimal numbers; others used a fractions multiplication algorithm)
- Students who multiplied and then spoke about estimating the product
- Students who explained their answers only by means of estimation

When we observe and analyze these groups of explanations, we can see a significant difference between students' levels of abstraction, levels of explanation (or proof in the advanced tasks), levels of experience with new principles or new concepts, and language the student understood and used.

Subsequently, a refined expert analysis was performed (Sabar Ben-Yehoshua, 1999). Finally, I formulated the final version of the theoretical structure comprising the types of answer expected from learners at a certain level, as well as characterization of the arithmetic thinking of learners at that level. Formulation of the levels was done following the research of the pre-service teachers but its aim was to be general and suit every person who studies arithmetic. The structure is presented in Table 1.

Findings

The present study explored the progress of students on the scale of levels following attendance of an arithmetic course (or a parallel one) in four different Israeli colleges. The results were presented by the levels of arithmetic thinking of PMTs planning to work in elementary school.

Analysis of Students' Explanations

The qualitative analysis of the arguments presented by the students focused on finding answers characteristic of learners at a certain level. In order to discern the answers typical of learners at the first level, I analyzed the arguments given

Table 1 Characteristics of levels of arithmetic thinking development

The level of arithmetic thinking development	Characteristics of answers given by learners who are at this level
Level 1: the performance level	Learners at this level are familiar with different numbers at the level of identification and can perform operations on them. These learners are still unfamiliar with the properties of the numbers and of the four arithmetic operations. Their solution methods are characterized by inefficient calculations and, generally speaking, their calculation competence is low. Verbalization of these learners is poor and their language is far from fluent and coherent. They stood out as learners who rely on instrumental comprehension (Skemp, 1976) of arithmetic.
Level 2: the explanatory level	Learners at this level are still at the stage of learning methods of writing numbers from different types (decimal system, rational number systems, etc.). They can compare numbers of same and of different types when they are given specific numbers; they are acquainted with properties of numbers and operations but cannot connect different arithmetic operations to their different properties. They can explain their assertions as assertions in general by means of an individual example or a reason by performing an "exercise." Nevertheless, there is an obvious gap between their performance capability and the difficulty in verbalization. The arithmetic terminology of these learners is partial and deficient and their ability to achieve generalization is only partial.
Level 3: the level of informal arithmetic	At this level, learners can connect the properties of the numbers to the properties of arithmetic operations, provided they are directly asked to do so. Learners are able to present informal arguments and support them by means of a generic example, partially using algebraic tools and so on. Moreover, learners at this level are capable of following deductive arguments and even provide some of these arguments. Yet, they are still unable to support the relations between arithmetic operations.
Level 4: the level of formal arithmetic	Learners at this level understand the logical need to support their mathematical conclusions. They are capable of analyzing an arithmetic claim, determining what is given and for which group of numbers this claim is correct. These learners can comprehend logical relations between the givens and can present a proof which is more or less formal (sometimes this type of proof combines reliance on a generic example or the proof itself is incomplete). At this level, learners begin using the central concepts in building the mathematical theory: claim, definition, theorem, proof, and so on. However, sometimes the meaning of these concepts is not entirely clear to the learners who are quite confused.

for questions 1–5 by the students classified as being at the first level of thinking. To discern the answers characteristic of learners at the second level of thinking, I analyzed the arguments given for questions 6–10 by the students classified at the second level of thinking, and so on. Below are the students' responses to one out of the five questions for each level of arithmetic thinking.

Level 1

To show characteristic answers of learners at this level, I chose item 5 (see Appendix). In most cases, the students explained this question by giving a full solution of an exercise, sometimes even in two different ways, but making no attempt to use estimation (see Fig. 1)

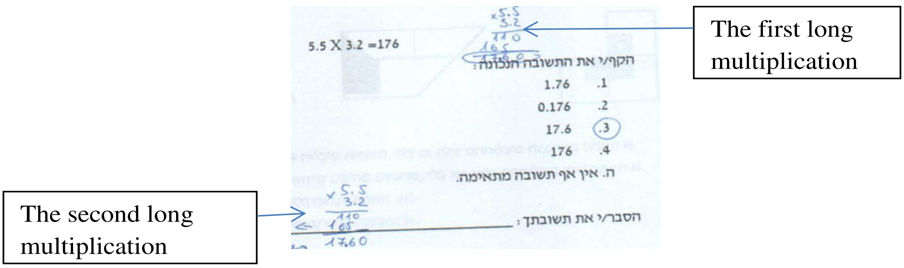

Fig. 1 Example of an argument given to item 5

Some of the subjects presented another explanation to the calculation which they believed should reinforce their choice. For example, one student wrote: "I am counting two places after obtaining the result." This argument indicates that the student knows what he/she should do but uses the wrong terminology: one does not count places but digits. One more example was given by another student: "3.2 × 5.5 = 17.60 and now, we can erase the remainder 0." With this argument, the student executes the algorithm correctly, but here too, his terminology fails him: he reduces the number by 10 by canceling the 0 as a result of referring to the 0 as a remainder and to the operation itself as a reduction. These and many other quotations facilitate identification of arithmetic behavior characteristics of the learners at this level: their ability to verbalize the execution is very poor and their arithmetic terminology is deficient. This characteristic can be found also in arguments given to other questions by students at the first level of thinking.

The arguments presented above are for a correct solution to the question. Below are some arguments for an erroneous solution to this question. The first recurrent mistake was reverse counting—counting from left to right, as exemplified, in the following arguments: "counting how many digits there are before the decimal point;" "moving the decimal point according to the distance of the product."

Regarding this question, it is important to point out that none of the students at this level relied on estimation, unlike some students at the higher levels. This indicates lack of mastery of the numbers' properties or inability to use these properties even when the learners are acquainted with them.

Level 2

This level of thinking is illustrated by students' responses to item 10 (see Appendix). The arguments given to this question can be classified into four categories. The first consists of arguments which show difficulty in providing an explanation. For example: "It is hard for me to explain." The students who applied this argument chose the correct distractor, but failed to explain their choice. It may be assumed, then, that students find it difficult to provide arguments, although these arguments were given prior to the course. Thus, it is logical to assume that this inability stems from the fact that the students have not engaged in arithmetic topics for a long period of time.

Another category consists of very short arguments, such as "You add $1\frac{1}{2}+3\frac{2}{4}=5$".

It is impossible to understand from this argument whether it stems from comprehension of the question or from a partial understanding of a situation whereby a certain number is subtracted from the first addend and another number is added to the second addend. In any case, the argument does not relate to the items themselves. Many of the students at the second level presented a similar argument. Thus, another characteristic of learners at the second level is giving partial explanations.

Some students gave the following argument, which indicates that they understood the question. For example: "We add what has been subtracted and what will be added." Nevertheless, these arguments are partial without general reference to the question. They are operational, describe what is being done, and sometimes include mistakes.

Another category comprises arguments by means of an individual example: 2 + 5 = 7; $\frac{1}{2} + \_\_\_ = 10\frac{2}{4}$. I arrived at the solution by means of an example. I chose 2 and 5. I decreased the 2 by $1\frac{1}{2}$ and the 7 by $3\frac{2}{4}$ and finally obtained the result of 10, namely from 5, I added 5 in order to get 10.

Level 3

Item 12 of the questionnaire was chosen to represent the arguments provided by students at level 3. The most effective way to know how many times a natural number 2 can be subtracted from natural number *b* is by division. It should be pointed out that most of the students at level 3 managed to "translate" the question into terms of the division operation. For instance, "Division because it saves us time and so we can know how many times *a* is included in *b*". These students relied on the claim that "how many times one can subtract" is identical in this case to the claim "how many times is included." Therefore, they determined that the most effective operation in this case is division. However, none of the students formally indicated the relation between subtraction and division. In other words, students at the third level are able to present informal arguments. This fact illustrates the inability of learners at this level to base their arguments on relations between arithmetic operations.

It is important to mention that some learners are at the third level according to the criteria. However, in fact, they failed to connect two arithmetic operations: "Some numbers cannot be divided and we want to divide and know exactly how many times". This argument shows that these students know that division is not a closed operation on natural numbers. The students translate the expression "how many times one can subtract" into the expression "how many times exactly number b comprises number a". According to the students, the meaning of this expression is an "exact" division without a remainder, something which is not always possible (where natural numbers are concerned). The students think that subtraction will solve the problem, since in this case, there will be no remainder. These students fail to cope with analysis of properties of the numbers system, namely examining the relations between the different arithmetic operations as well as the relations between these operations and different types of numbers. Before administration of the questionnaire, the assumption was that learners at the third level of arithmetic thinking development understand the relations between different arithmetic operations. However, in the course of the study this was not corroborated. Some learners acted in other cases according to this expectation, but not in this case. This problem might result from lack of learning or lack of reference to relations of this type in their previous studies.

Level 4

The question chosen as representative of the arguments given by the students at the fourth level is associated with the properties of 0 in division (item 19). The arguments provided for this question can be classified into two categories: arguments given prior to the course and arguments given following the course. According to the earlier arguments, the proof that this division contradicts arithmetic operations is founded on one individual example. The student assumes that when dividing by 0, the quotient should be 0 and proves that this is wrong. Hence, he concludes that dividing by 0 is impossible: "Since it is impossible to divide by 0 and this is neither a definition nor an axiom. This proves that such an exercise cannot be executed. For example, 4 : 0 = 0—in order to check whether this exercise is correct, we should use multiplication: 0 × 0 = [not 4] 0 and therefore division by 0 cannot be executed." Here, the student presents all the transitions necessary for a deductive proof. He frequently uses verbal representation and there is also a numerical representation. On the other hand, the symbolic representation is not included. This proof is based on the generic example (Balacheff, 1988). Most of the arguments offered after the course attested to the learners' ability to identify logical relations between the data and perform a claim which is more or less formal as required. For example: "Let's assume $\frac{7}{0} = x$, multiplication and division are defined as inverse operations, namely c: a = b and $a \cdot b = c \Leftrightarrow c : b = a$ and then 7 = x · 0 and there is no number which will provide a correct expression".

General Data Obtained from Analyzing the Questionnaire

First, I will display the students' distribution according to the levels of arithmetic thinking development.

Table 2 Distribution of students according to levels of arithmetic thinking development

Level	Frequency	
	Pre-test	Post-test
0	17 (18 %)	6 (6 %)
1	14 (15 %)	12 (13 %)
2	29 (30 %)	37 (40 %)
3	18 (20 %)	11 (12 %)
4	18 (19 %)	28 (30 %)
Total subjects at all levels	96	94

As Table 2 demonstrates, there are students who are not even at the first level of arithmetic thinking development. These students failed to respond correctly to four out of the first five items of the questionnaire. These findings are in line with Senk (1989) and Clements & Battista (1992) in their studies of the development of geometric thinking. Clements & Battista defined this level of geometry and named it as level 0 (pre-cognitive level). Similarly, in the present study, this level is defined as level 0 and the reasons provided by these students have several common characteristics:

- Using indefinite statements which attest to allegedly conceptual thinking. For example: every number has a digit of units, of tens and of hundreds
- Providing an argument which has no relation to the question given. For example: this question cannot be logically solved
- In most cases, these students did not give any reasons or they explained their choice by expressions, such as: “I am not sure,” “It seems to me,” and others

Test χ^2 shows significant changes in the students’ levels of thinking following the course ($\chi^2 = 10.23$; $df = 4$; $p < 0.05$). The most prominent changes were at levels 4 and 2; whereby, the percentage of those responding correctly in the post-test increased (11 and 9 %, respectively). In the same way, there was a significant decrease in the percentage of students classified at level 0 in the pre-test versus the percentage classified at this level in the post-test (from 18 to 7 %). At level 1, there were hardly any changes in the number of students between the two tests (14 and 13 %, respectively). At level 3, there was a decrease in the percentage of students classified at this level in the post-test (19 and 12 %, respectively). This decrease can be attributed to the movement in the number of students at levels 2 and 4.

Moreover, the findings indicate that the percentage of students classified at low levels of arithmetic thinking (approximately 43 % at levels 0–1) considerably decreased in the post-test (approximately 19 % at levels 0–1) vis-à-vis higher levels of thinking. However, even after a year-long course, there were students whose level of arithmetic thinking remained at level 0. Perusal of the pre-test results leads to the conclusion that many students (about 1/3) are at level 2, with an almost equal distribution between the low levels (0–2)—approximately 32 % and the high levels (3–4)—approximately 38 %. Results of the post-test illustrate that there are many more students at Level 2 (about 40 %) and, in addition, more students at the high levels (40 %) compared to 18 % at the low levels.

The table also shows that the major progress is from level 0 to 1 and from level 1 to 2. Furthermore, it can be assumed that the progress is by one level. It is interesting to examine how many progressed from levels 1 to 2, etc. It is also interesting to see whether there were leaps of levels. I was unable to check this since the questionnaires were completed anonymously. Hence, the assumption that can be presented here (but which cannot be explored within the framework of the present study) is that most of the students’ progress was by one level only, with only a few exceptions. Considering the research limitations of the present study, one can stipulate this with a high degree of probability.

Discussion and Conclusions

The present study explored the development of arithmetic thinking of pre-service teachers during an arithmetic course in their teacher education program. In this study, I analyzed the development of arithmetic thinking on the basis of van Hiele's theory of the development of mathematical thinking as expressed in PMT's ability to solve 20 assignments that divide into four levels of thinking, by analysis of PMTs' correct solution and explanation for each of these assignments. Development of pre-service teachers' arithmetic thinking was examined by analysis of change in the level of arithmetic thinking from the start to completion of arithmetic course. In line with studies on the geometric knowledge of PMTs which demonstrated that the development of this knowledge to the level of conceptual understanding can be described in terms of progression of geometric thinking (e.g. Battista, 2007), this study demonstrated that referring to the development of arithmetic thinking is the effective tool for evaluation of the effectiveness of an arithmetic course.

Levels of Arithmetic Thinking

The theory of the development of arithmetic thinking according to levels enables the claim that there are students with different and clearly distinct levels of arithmetic thinking. The differences between levels are manifested by the students' mathematical language, their way of arguing and justifying claims, and the tools they use for making mathematical judgments and decisions. Based on the Guttman Scale Analysis, it can be determined that the levels are hierarchical. Moreover, there is a basis for the suggestion that the levels of arithmetic thinking development do not depend on biological age, since various levels of thinking were identified among the PMTs. A question that arises following the findings is: Is this distinction between one level and another so important?

Analysis of findings obtained in the present study shows that after the course, more than half the PMTs are still at levels 0–2. An additional finding is that the students' progress as a result from participation in the arithmetic course, is mainly by one level. That is to say, the number of students whose arithmetic thinking can be characterized as low remained higher than expected. Based on the characterization of levels of arithmetic reasoning in this paper, this means that relationships and connections between various groups of numbers, between various arithmetic operations, and between properties of numbers and properties of operations, were not acquired by a large part of the students who participated in the course. These findings indicate that PMTs don't acquire a conceptual understanding of arithmetic (Hiebert & Lefevre, 1986) and don't enhance the knowledge required of mathematics teachers. As is clear from the research, this expectation from teachers who will teach mathematics in elementary school is not realized for a large number of PMTs as a result of the arithmetic course. The tool of levels of arithmetic thinking allows the observation of the PMTs' education from a new angle regarding the impact of an arithmetic course on the students' development of arithmetic thinking as discussed below.

The Way to Design the Arithmetic Course for PMTs

This study aimed to suggest a way to design an appropriate arithmetic course for PMTs. As described above, this course must match the level of arithmetic thinking of the PMTs and their level of teaching. The main reason for this is that the findings of the study allow us to assume that people at different levels are unable to understand each other. They talk, in fact, in two "foreign languages." This occurs in teaching when teachers do not consider the level of thinking development of their pupils. However, teaching then complies with the learners' level—not too high or too low for them; teachers avoid creating a communication gap between themselves and their pupils and prevent the formation of a cognitive barrier between themselves and their pupils. It is therefore important to pay attention to issues such as the types of reasoning that students give to their answers, the language they use, and how they justify their claims. These findings have led to the conclusion that the development of arithmetic thinking can be identified by the level of explanation that the student gives. Moreover, when learners progress from one level to another, this language is expanded and enriched. This finding is similar to the findings from studies conducted in the development of geometric thinking (Clements & Battista, 1992). An additional aspect worth relating to when constructing an arithmetic course is the desired level of the development of arithmetic thinking among PMTs. The decision about this desired level should ensue from what is required of the elementary school mathematics teacher. As it emerges for a review of the literature, the required mathematical knowledge of teachers must be deep and broad (Ball et al., 2008; Ma, 1999). The teacher must understand elementary mathematics in order to know not only what, but also why something works. This means that the teacher must understand the hidden meanings according to justified ideas and procedures learned, and be able to make connections and links between concepts and their properties. Accordingly, and as this study shows, PMTs must reach at least level 3 on the arithmetic thinking development scale—that of informal arithmetic.

In addition to the study of arithmetic topics relevant to the elementary school curriculum, this course needs to be more focused on the explanation and representation of connectedness and relationships between kinds of numbers, between numbers and operations and between different operations, and between properties of numbers and properties of operations. From the analysis of the results of this study and many studies on the teaching of geometry, it seems that this approach may lead to higher quality learning.

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